

HEAT CONTENT ASYMPTOTICS WITH SINGULAR DATA

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ABSTRACT. We study the asymptotic behaviour of the heat content on a compact Riemannian manifold with boundary and with singular specific heat and singular initial temperature distributions. Assuming the existence of a complete asymptotic series we determine the first three terms in that series. In addition to the general setting, the interval is studied in detail.

Keywords: Heat content, compact Riemannian manifold, singular data

Classification: 58J32; 58J35; 35K20

DEDICATION

This paper is dedicated to Stewart Dowker. Stuart has made many contributions to the field of spectral geometry and remains active in this area [14, 15]. The second author has been honored to have been a collaborator with Stewart [16, 17]. We hope this paper serves as a fitting tribute to our colleague and friend.

1. INTRODUCTION

Let M be a compact Riemannian manifold with smooth boundary ∂M , and let δ denote the geodesic distance to the boundary. Let ψ_1 and ψ_2 be smooth functions on the interior of M . Then ψ_1 will represent the initial temperature of M and ψ_2 will represent the specific heat of M . Since M is compact and ∂M is smooth the distance function is smooth near ∂M . We have to assume that $\delta^{\alpha_1}\psi_1$ and $\delta^{\alpha_2}\psi_2$ are smooth on a closed collared neighbourhood of ∂M . The parameters α_1 and α_2 control the growth or decay of ψ_1 and ψ_2 near ∂M . Let D be an operator of Laplace type on M . Impose Dirichlet boundary conditions to define the realization of D . Let dx be the Riemannian measure on M . Since ∂M is smooth, the corresponding Dirichlet heat kernel $p_M(x_1, x_2; t)$, $x_1 \in M, x_2 \in M, t > 0$ vanishes linearly in $\delta(x_1)$ and $\delta(x_2)$ near ∂M . We suppose $\alpha_1 < 2$ and $\alpha_2 < 2$ to ensure convergence subsequently. Let e^{-tD} be the fundamental solution of the heat equation for the Dirichlet Laplacian. Then

$$u_1(\cdot, t) := e^{-tD}\psi_1 = \int_M p_M(\cdot, x; t)\psi_1(x)dx$$

represents the temperature of the manifold for $t > 0$. The *heat content* Q is defined by

$$\begin{aligned} Q(\psi_1, \psi_2, D)(t) : &= \int_M u_1(x; t) \cdot \psi_2(x)dx \\ &= \int_M \int_M p_M(x_1, x_2; t)\psi_1(x_1)\psi_2(x_2)dx_1dx_2. \end{aligned}$$

Conjecture 1. *Let $\alpha_1 + \alpha_2 \notin \mathbb{Z}$, $\alpha_1 < 2$, $\alpha_2 < 2$. There is a complete asymptotic series as $t \downarrow 0$*

$$Q(\psi_1, \psi_2, D)(t) \sim \sum_{n=0}^{\infty} t^n \beta_n^M + \sum_{j=0}^{\infty} t^{(1+j-\alpha_1-\alpha_2)/2} \beta_j^{\partial M},$$

where the $\beta_n^M, n = 0, 1, \dots$ are regularized integrals of local invariants over M and where the $\beta_j^{\partial M}, j = 0, 1, \dots$ are integrals of local invariants over the boundary.

Remark 1. It is convenient to let α_1 and α_2 be complex as we may then use analytic continuation. For $\Re(\alpha_1) < 0$ and $\Re(\alpha_2) < 0$,

$$\beta_n^M = (-1)^n \frac{1}{n!} \int_M D^n \psi_1(x) \cdot \psi_2(x) dx.$$

The values of β_n^M for more general values of α_1 and α_2 may then be obtained as regularized integrals as discussed in [10]. We omit the technical details concerning the requisite regularizations in the interests of brevity as they will play no role in our analysis.

The heat content has obvious physical relevance and the invariants $\beta_j^{\partial M}$, which reflect the asymptotic behaviour as $t \downarrow 0$, relate the geometry of M to the underlying physical properties of M . Much of the previous work in the field has been devoted to the computation of the invariants $\beta_j^{\partial M}$ in the smooth setting ($\alpha_1 = 0, \alpha_2 = 0$). They were originally studied for the scalar Laplacian with $\psi_1 = \psi_2 = 1$ [2, 4, 11]. Subsequently, general initial temperatures and specific heats were investigated – see [5, 10, 13, 19, 20, 21, 22] and the references contained therein. Other boundary conditions (Neumann, Zaremba, etc.) have been considered [3, 8]. The growth of the coefficients $\beta_j^{\partial M}$ has also been of interest [1, 7, 23] – see also [12] for related work on the heat trace asymptotics. The case where ψ_1 is singular ($\alpha_1 > 0$) but ψ_2 is smooth ($\alpha_2 = 0$) was studied previously [9, 10]. The current paper is devoted to the study of the invariants $\beta_j^{\partial M}$ in the doubly singular case.

The special case of a ball of radius a in \mathbb{R}^3 is well understood. The following result was proved in [6].

Theorem 1. Let $B_a = \{x \in \mathbb{R}^3 : |x| \leq a\}$, and let D be the Dirichlet Laplacian acting in $L^2(B_a)$. If $\alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 > 3, J \in \mathbb{N}$ then there exist coefficients b_0, b_1, \dots depending on α_1, α_2 only such that for $t \downarrow 0$

$$\begin{aligned} Q(\delta^{-\alpha_1}, \delta^{-\alpha_2}, D)(t) &= 4\pi c_{\alpha_1, \alpha_2} a^2 t^{(1-\alpha_1-\alpha_2)/2} - 4\pi(c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1}) a t^{(2-\alpha_1-\alpha_2)/2} \\ &\quad + 4\pi c_{\alpha_1-1, \alpha_2-1} t^{(3-\alpha_1-\alpha_2)/2} + \sum_{j=0}^J b_j a^{3-j-\alpha_1-\alpha_2} t^{j/2} + O(t^{(J+1)/2}), \end{aligned} \quad (1)$$

where

$$\begin{aligned} c_{\alpha_1, \alpha_2} &= 2^{-\alpha_1-\alpha_2} \pi^{-1/2} \Gamma((2-\alpha_1-\alpha_2)/2) \\ &\quad \times \int_0^1 (\rho^{-\alpha_1} + \rho^{-\alpha_2}) ((1-\rho)^{\alpha_1+\alpha_2-2} - (1+\rho)^{\alpha_1+\alpha_2-2}) d\rho, \end{aligned} \quad (2)$$

and

$$\begin{aligned} b_0 &= -8\pi((\alpha_1 + \alpha_2 - 1)(\alpha_1 + \alpha_2 - 2)(\alpha_1 + \alpha_2 - 3))^{-1}, & b_1 &= 0, \\ b_2 &= 8\pi\alpha_1\alpha_2((\alpha_1 + \alpha_2 + 1)(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 - 1))^{-1}, & b_3 &= 0. \end{aligned}$$

It is convenient to use a standard formalism to describe the invariants $\beta_j^{\partial M}$ in the general setting. Let D be an operator of Laplace type acting on the space of smooth sections to some vector bundle V over a Riemannian manifold (M, g) . Choose a local system of coordinates (x^1, \dots, x^m) for M and a local frame for V . We adopt the *Einstein convention* and sum over repeated indices. Let $ds^2 = g_{\mu\nu} dx^\mu \circ dx^\nu$ define the Riemannian metric and let $g^{\mu\nu}$ be the inverse matrix where $1 \leq \mu, \nu \leq m$. We may then express:

$$D = -(g^{\mu\nu} \text{Id} \partial_{x_\mu} \partial_{x_\nu} + A^\mu \partial_\mu + B)$$

for suitably chosen endomorphisms A^μ and B of V . If ∇ is a connection on V , we use ∇ and the Levi-Civita connection to covariantly differentiate tensors of all types and let ‘;’ denote multiple covariant differentiation. If ψ_1 is a section to V which is smooth on $\text{int}(M)$, let $\psi_{1;\mu\nu}$ be the components of $\nabla^2\psi_1$. If E is an auxiliary endomorphism of V , we define the associated *modified Bochner Laplacian* by setting:

$$D(g, \nabla, E)\psi_1 := -g^{\mu\nu}\psi_{1;\nu\mu} - E\psi_1.$$

Let $\Gamma_{\mu\nu\sigma}$ and $\Gamma_{\mu\nu}{}^\sigma$ be the Christoffel symbols of the Levi-Civita connection. Then (see, for example, the discussion in [5]):

Lemma 2. *If D is an operator of Laplace type, then there exists a unique connection ∇ on V and a unique endomorphism E of V so that $D = D(g, \nabla, E)$. The connection 1-form ω of ∇ and the endomorphism E are given by*

$$\begin{aligned}\omega_\mu &= \frac{1}{2}(g_{\mu\nu}A^\nu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon\mu}\text{Id}), \\ E &= B - g^{\mu\nu}(\partial_{x_\nu}\omega_\mu + \omega_\mu\omega_\nu - \omega_\sigma\Gamma_{\mu\nu}{}^\sigma).\end{aligned}$$

The specific heat ψ_2 is a section to the dual vector bundle \tilde{V} . We use the dual connection on \tilde{V} to covariantly differentiate ψ_2 . Note that the connection 1 form $\tilde{\omega}_\nu$ for $\tilde{\nabla}$ is the dual of $-\omega_\nu$. Thus

$$\tilde{\nabla}_{\partial_{x_\mu}} = \partial_{x_\mu} - \frac{1}{2}(g_{\mu\nu}\tilde{A}^\nu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon\mu}\text{Id}).$$

Near the boundary, choose an orthonormal frame $\{e_1, \dots, e_m\}$ for the tangent bundle of M so that e_m is the inward unit geodesic normal. Let indices a, b range from 1 to $m-1$ and index the induced orthonormal frame $\{e_1, \dots, e_{m-1}\}$ for the tangent bundle of the boundary. We let ‘.’ denote the components of tangential covariant differentiation defined by ∇ and the Levi-Civita connection of the boundary. Let $L_{ab} := g(\nabla_{e_a}e_b, e_m) = \Gamma_{abm}$ be the components of the second fundamental form. The difference between ‘;’ and ‘.’ is then measured by L . Let \tilde{D} be the dual operator of Laplace type on \tilde{V} . The following relations will be useful subsequently:

$$\begin{aligned}D\psi_1 &= -(\psi_{1:aa} + \psi_{1:mm} - L_{aa}\psi_{1;m} + E\psi_1), \\ \tilde{D}\psi_2 &= -(\psi_{2:aa} + \psi_{2:mm} - L_{aa}\psi_{2;m} + \tilde{E}\psi_2).\end{aligned}$$

We expand ψ_1 and ψ_2 near the boundary of M in the form:

$$\psi_1(y, \delta) \sim \delta^{-\alpha_1} \sum_{j=0}^{\infty} \psi_1^j \delta^j, \quad \psi_2(y, \delta) \sim \delta^{-\alpha_2} \sum_{j=0}^{\infty} \psi_2^j \delta^j,$$

where $\nabla_{e_m}\psi_1^j = 0$ and $\tilde{\nabla}_{e_m}\psi_2^j = 0$. We shall usually be working with scalar operators and can choose local sections s and \tilde{s} so that $\nabla_{e_m}s = 0$ and $\tilde{\nabla}_{e_m}\tilde{s} = 0$. We may then express $\psi_1 = \Psi_1 s$, $\psi_1^i = \Psi_1^i s$, $\psi_2 = \Psi_2 \tilde{s}$, and $\psi_2^i = \Psi_2^i \tilde{s}$ where Ψ_1^j and Ψ_2^j are smooth functions defined on the boundary so that we have the modified Taylor series

$$\delta^{\alpha_i} \Psi_i(y, \delta) \sim \sum_{j=0}^{\infty} \Psi_i^j(y) \delta^j \quad \text{for } i = 1, 2.$$

For the Laplacian, the bundles and connections under consideration are trivial so this formalism is unnecessary. However, for more general operators, the connections in question are not flat and this formalism is essential. We shall be using the method of “universal examples” in what follows. It is a peculiar feature of this method that even if we were only interested in the scalar Laplacian for a smooth bounded domain in \mathbb{R}^m , it would be necessary to deal with quite general operators as we shall see presently while proving Lemma 11 in Section 3.

Let Ric be the Ricci tensor of M , let τ be the scalar curvature of M , and let dy be the Riemannian measure of ∂M . Section 3 is devoted to the proof of the following result:

Theorem 3. *Let $\alpha_1 + \alpha_2 \notin \mathbb{Z}$, $\alpha_1 < 2$, $\alpha_2 < 2$. Assume that Conjecture 1 holds. Let c_{α_1, α_2} be as given in Equation (2). Then*

$$\begin{aligned}\beta_0^{\partial M} &= \int_{\partial M} c_{\alpha_1, \alpha_2} \psi_1^0 \psi_2^0 dy, \\ \beta_1^{\partial M} &= \int_{\partial M} \{c_{\alpha_1-1, \alpha_2} \psi_1^1 \psi_2^0 - \frac{1}{2} \{c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1}\} \psi_1^0 \psi_2^0 L_{aa} \\ &\quad + c_{\alpha_1, \alpha_2-1} \psi_1^0 \psi_2^1\} dy, \\ \beta_2^{\partial M} &= \int_{\partial M} \{c_{\alpha_1-2, \alpha_2} \psi_1^2 \psi_2^0 - \frac{1}{2} (c_{\alpha_1-2, \alpha_2} + c_{\alpha_1-1, \alpha_2-1}) L_{aa} \psi_1^1 \psi_2^0 \\ &\quad + c_{\alpha_1, \alpha_2} E \psi_1^0 \psi_2^0 + c_{\alpha_1, \alpha_2-2} \psi_1^0 \psi_2^2 - \frac{1}{2} (c_{\alpha_1-1, \alpha_2-1} + c_{\alpha_1, \alpha_2-2}) L_{aa} \psi_1^0 \psi_2^1 \\ &\quad + (-\frac{1}{4} c_{\alpha_1-2, \alpha_2} - \frac{1}{4} c_{\alpha_1, \alpha_2-2} + \frac{1}{2} c_{\alpha_1, \alpha_2}) (L_{aa} L_{bb} + \text{Ric}_{mm}) \psi_1^0 \psi_2^0 \\ &\quad - c_{\alpha_1, \alpha_2} \psi_1^0 : a \rho_{20:a}^0 + 0 \tau \psi_1^0 \psi_2^0 + c_{\alpha_1-1, \alpha_2-1} \psi_1^1 \psi_2^1 \\ &\quad + (\frac{1}{8} c_{\alpha_1-2, \alpha_2} + \frac{1}{8} c_{\alpha_1, \alpha_2-2} + \frac{1}{4} c_{\alpha_1-1, \alpha_2-1} - \frac{1}{4} c_{\alpha_1, \alpha_2}) L_{aa} L_{bb} \psi_1^0 \psi_2^0\} dy.\end{aligned}$$

For the ball B_a in \mathbb{R}^3 , $L_{aa} = 2a^{-1}$, $L_{aa} L_{bb} = 4a^{-2}$ and $L_{aa} L_{bb} = 2a^{-2}$. Hence

$$\beta_1^{\partial B_a} = -4\pi a (c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1}),$$

and the coefficient of $t^{(2-\alpha_1-\alpha_2)/2}$ agrees with Equation (1) in Theorem 3. Similarly, the next term in the series given by $\beta_2^{\partial B_a}$ is consistent with Theorem 3.

We observe that the Γ -function in the expression for c_{α_1, α_2} which is given in Equation (2) has simple poles for $\alpha_1 + \alpha_2 \in \{2, 4, 6, \dots\}$. Furthermore the integrand with respect to ρ equals 0 for $\alpha_1 + \alpha_2 = 2$. It is easily seen that this singularity is removable. On the other hand the integral with respect to ρ is finite only for $\alpha_1 < 2, \alpha_2 < 2$ and $\alpha_1 + \alpha_2 > 1$. This suggests that the j^{th} term ($j = 1, 2, 3$) in Equation (1) will take a different form for $\alpha_1 + \alpha_2 = j$. This is indeed the case for an interval in \mathbb{R} . Let $a > 0$, and let χ_1, χ_2 be non-negative C^∞ functions on \mathbb{R}^+ defined by

$$\chi_{1,2}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \epsilon_{1,2}, \\ 0 & \text{if } x \geq \epsilon_{3,4}, \end{cases}$$

where $0 < \epsilon_1 < \epsilon_3 < a/2$ and $0 < \epsilon_2 < \epsilon_4 < a/2$. We shall establish the following result in Section 2:

Theorem 4. *Let $\alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 = 1$. If $t \downarrow 0$, then:*

$$\begin{aligned}& \int \int_{[0,a]^2} p_{[0,a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} dx_1 dx_2 \\ &= \log(\epsilon^2/t) + \gamma + 4 \log(2^{1/2} - 1) + 4 \log 2 \\ &\quad + 2 \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx + \int_{[0,1]} dq q^{-1} (1 + q^2) \\ &\quad \times \left\{ ((1+q)/(1-q))^{\alpha-1} + ((1-q)/(1+q))^\alpha \right. \\ &\quad \left. - 2(1-q)(1+q^2)^{-1/2} \right\} + O(t^{1/2} \log t),\end{aligned}\tag{3}$$

where $p_{[0,a]}(x_1, x_2; t), x_1 \in [0, a], x_2 \in [0, a], t > 0$ is the Dirichlet heat kernel for the interval $[0, a]$, γ is Euler's constant, and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

We note that the ϵ -dependence in the right hand side of Equation (3) is fictitious. Since $\chi_1(x) = \chi_2(x) = 1$ for $0 < x \leq \epsilon$, we have that

$$\log(\epsilon^2/t) + 2 \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx = 2 \int_{[\sqrt{t}, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx,$$

which independent of ϵ for $0 < t < \epsilon^2$. We also note that the leading term in Theorem 4 jibes with Theorem 1.4 (2) in [10] since the volume of the volume of the boundary of the interval $[0, a]$ is equal to 2. This supports the following.

Conjecture 2. *Let M be a compact Riemannian manifold with smooth boundary ∂M , and let δ denote the distance to the boundary. Let $\alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 = 1$, and let χ_1 and χ_2 be smooth functions on \mathbb{R}_+ with support contained in an interval $[0, b]$, and equal to 1 in a neighbourhood of 0, and where b is such that δ is smooth on the collar $\partial M \times [0, b]$. If $t \downarrow 0$ then*

$$Q(\delta^{-\alpha_1} \chi_1 \circ \delta, \delta^{-\alpha_2} \chi_2 \circ \delta, D)(t) = 2^{-1} \int_{\partial M} dy \log t + o(\log t).$$

We note that Theorem 4 and Conjecture 2 include the cases where either $1 < \alpha_1 < 2$ or $1 < \alpha_2 < 2$. This requires more care in the proof of Theorem 4 than the case where both $\alpha_1 < 1$ and $\alpha_2 < 1$.

2. THE PROOF OF THEOREM 4

The first step in the proof of Theorem 4 is to reduce the calculation on the interval $[0, a]$ to a calculation on the half-line $\mathbb{R}^+ = [0, \infty)$. We have the following:

Lemma 5. *Let $\alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 = 1$. If $t \downarrow 0$ then*

$$\begin{aligned} & \int \int_{[0, a]^2} p_{[0, a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} dx_1 dx_2 \\ &= 2 \int \int_{\mathbb{R}_+^2} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2 \\ & \quad + O(e^{-(1-\eta)\kappa^2/(4t)}), \end{aligned} \tag{4}$$

where $\kappa = a - \epsilon_3 - \epsilon_4$ and $\eta = \max\{\alpha_1/2, \alpha_2/2\}$.

Proof. Without loss of generality we may assume that $\alpha_1 \geq \alpha_2$. We partition the region of integration $[0, a]^2 = \cup_{i=1}^5 A_i$, where

$$\begin{aligned} A_1 &= [0, \epsilon_3] \times [0, \epsilon_4], & A_2 &= [0, \epsilon_3] \times [a - \epsilon_4, a], \\ A_3 &= [a - \epsilon_3, a] \times [0, \epsilon_4], & A_4 &= [a - \epsilon_3, a] \times [a - \epsilon_4, a], \\ A_5 &= A \setminus (\cup_{i=1}^4 A_i). \end{aligned}$$

The integrand in the left hand side of Equation (4) is identically equal to 0 on A_5 , and this set does not contribute to the integral. Since

$$\begin{aligned} p_{[0, a]}(x_1, x_2; t) &= p_{[0, a]}(a - x_1, a - x_2; t), \quad \text{and} \\ p_{[0, a]}(x_1, a - x_2; t) &= p_{[0, a]}(a - x_1, x_2; t), \end{aligned}$$

the contributions of A_1 and A_2 to the integral in the left hand side of Equation (4) are equal to the contributions of A_4 and A_3 respectively. Since $|x_1 - x_2| \geq \kappa$ for $(x_1, x_2) \in A_2$, we have by monotonicity of the Dirichlet heat kernel that

$$\begin{aligned} p_{[0, a]}(x_1, x_2; t) &\leq p_{\mathbb{R}^+}(x_1, x_2; t) \\ &= (4\pi t)^{-1/2} \left(e^{-(x_1 - x_2)^2/(4t)} - e^{-(x_1 + x_2)^2/(4t)} \right) \\ &= (4\pi t)^{-1/2} e^{-(x_1 - x_2)^2/(4t)} (1 - e^{-x_1 x_2/t}) \\ &\leq t^{-3/2} x_1 x_2 e^{-\kappa^2/(4t)}. \end{aligned} \tag{5}$$

Hence the contribution from A_2 to the integral in the left hand side of Equation (4) is bounded from above by

$$t^{-3/2} e^{-\kappa^2/(4t)} \int \int_{A_2} \chi_1(x_1) \chi_2(x_2) x_1^{1-\alpha_1} x_2^{\alpha_1} dx_1 dx_2 = O(e^{-\eta\kappa^2/(4t)}).$$

The contribution from A_1 to the integral in the left hand side of Equation (4) is bounded from above by

$$\begin{aligned} & \int \int_{A_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2 \\ &= \int \int_{\mathbb{R}_+^2} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2. \end{aligned}$$

This completes the proof of the upper bound.

To establish the lower bound we note that

$$\begin{aligned} & \int \int_{[0,a]^2} p_{[0,a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} dx_1 dx_2 \\ & \geq 2 \int \int_{A_1} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2. \end{aligned}$$

It is well known that the Dirichlet heat kernel for an open set $\Omega \in \mathbb{R}^m$ has the following probabilistic representation.

$$p_{\Omega}(x_1, x_2; t) = p_{\mathbb{R}^m}(x_1, x_2; t) \text{Prob}_{x_1, x_2}[B(s) \in \Omega, 0 < s < t],$$

where $(B(s), 0 \leq s \leq t)$ is a Brownian bridge on \mathbb{R}^m . For $\Omega = [0, a] \in \mathbb{R}$ and for $x \in [0, a], y \in [0, a]$ we have that

$$\begin{aligned} p_{[0,a]}(x_1, x_2; t) &= p_{\mathbb{R}}(x_1, x_2; t) \text{Prob}_{x_1, x_2}[0 < B(s) < a, 0 < s < t] \\ &= p_{\mathbb{R}}(x_1, x_2; t) (\text{Prob}_{x_1, x_2}[0 < B(s), 0 < s < t] \\ &\quad - \text{Prob}_{x_1, x_2}[(0 < B(s), 0 < s < t) \wedge (\max_{0 \leq s \leq t} B(s) \geq a)]). \end{aligned} \tag{6}$$

By Hölder's inequality we have for $\eta \in (0, 1)$

$$\begin{aligned} & p_{\mathbb{R}}(x_1, x_2; t) \text{Prob}_{x_1, x_2}[(0 < B(s), 0 < s < t) \wedge (\max_{0 \leq s \leq t} B(s) \geq a)] \\ & \leq p_{\mathbb{R}}(x_1, x_2; t) (\text{Prob}_{x_1, x_2}[0 < B(s), 0 < s < t])^{\eta} \\ & \quad \times (1 - \text{Prob}_{x_1, x_2}[B(s) < a, 0 < s < t])^{1-\eta} \\ & = (p_{\mathbb{R}^+}(x_1, x_2; t))^{\eta} (p_{\mathbb{R}}(x_1, x_2; t) - p_{(-\infty, a]}(x_1, x_2; t))^{1-\eta}. \end{aligned} \tag{7}$$

Since

$$p_{(-\infty, a]}(x_1, x_2; t) = (4\pi t)^{-1/2} \left(e^{-(x_1-x_2)^2/(4t)} - e^{-(2a-x_1-x_2)^2/(4t)} \right), \tag{8}$$

we have by Equation (6), Equation (7), and Equation (8) that

$$\begin{aligned} p_{[0,a]}(x_1, x_2; t) & \geq p_{\mathbb{R}^+}(x_1, x_2; t) \\ & \quad - (p_{\mathbb{R}^+}(x_1, x_2; t))^{\eta} (4\pi t)^{-(1-\eta)/2} e^{-(1-\eta)(2a-x_1-x_2)^2/(4t)}. \end{aligned}$$

By the last inequality in Equation (5)

$$\begin{aligned} & (p_{\mathbb{R}^+}(x_1, x_2; t))^{\eta} (4\pi t)^{-(1-\eta)/2} e^{-(1-\eta)(2a-x_1-x_2)^2/(4t)} \\ & \leq t^{-\frac{1}{2}-\eta} (x_1 x_2)^{\eta} e^{-(1-\eta)(2a-x_1-x_2)^2/(4t)}. \end{aligned}$$

Integrating the above right hand side with respect to $\chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2$ yields a bound

$$\begin{aligned} & t^{-\frac{1}{2}-\eta} \int_{[0, \epsilon_3]} \chi_1(x_1) x_1^{\eta-\alpha_1} dx_1 \int_{[0, \epsilon_4]} \chi_2(x_2) x_2^{\eta-\alpha_2} dx_2 e^{-(1-\eta)(2a-x_1-x_2)^2/(4t)} \\ & \leq t^{-\frac{1}{2}-\eta} e^{-(1-\eta)(2a-\epsilon_3-\epsilon_4)^2/(4t)} \int_{[0, \epsilon_3]} \chi_1(x_1) x_1^{\eta-\alpha_1} dx_1 \int_{[0, \epsilon_4]} \chi_2(x_2) x_2^{\eta-\alpha_2} dx_2 \\ & = O(e^{-(1-\eta)(a-\epsilon_3-\epsilon_4)^2/(4t)}). \end{aligned}$$

Note that since $2 > \alpha_1$ and $\eta = \alpha_1/2$, $x_1^{\eta-\alpha_1} = x_1^{-\alpha_1/2}$ is integrable at 0. Since $1 = \alpha_1 + \alpha_2 \leq 2\alpha_1$ we have that $\alpha_1 \geq 1/2 > 0$. Hence $x_2^{\eta-\alpha_2} = x_2^{-1+(3\alpha_1/2)}$ is also integrable at 0. This completes the proof of the lower bound. \square

In order to prove Theorem 4 it clearly suffices to prove the following.

Lemma 6. *Let $\alpha_2 \leq \alpha_1 < 2, \alpha_1 + \alpha_2 = 1$. If $t \downarrow 0$ then*

$$\begin{aligned} & \int \int_{\mathbb{R}_+^2} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2 \\ &= 2^{-1} \log(\epsilon^2/t) + 2^{-1} \gamma + 2 \log(2^{1/2} - 1) + 2 \log 2 \\ & \quad + \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx + 2^{-1} \int_{[0,1]} dq q^{-1} (1+q^2) \\ & \quad \times \left\{ ((1+q)/(1-q))^{\alpha-1} + ((1-q)/(1+q))^{\alpha} \right. \\ & \quad \left. - 2(1-q)(1+q^2)^{-1/2} \right\} + O(t^{1/2} \log t). \end{aligned} \quad (9)$$

Proof. Define

$$\begin{aligned} C &= \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1^2 + x_2^2 \geq \epsilon^2, 0 \leq x_1 \leq \epsilon_3, 0 \leq x_2 \leq \epsilon_4\}, \\ C_1 &= \{(x_1, x_2) \in C : |x_1 - x_2| \leq \sigma\}, \end{aligned}$$

where $\sigma \in (0, \epsilon/5)$ will be chosen later on. The left hand side of Equation (9) can be written as $B_1 + B_2$, where

$$\begin{aligned} B_1 &= \int \int_{\mathbb{R}_+^2 \cap \{x_1^2 + x_2^2 < \epsilon^2\}} p_{\mathbb{R}^+}(x_1, x_2; t) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2, \\ B_2 &= \int \int_C p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2. \end{aligned} \quad (10)$$

To estimate B_2 we first consider the contribution from the set $C \setminus C_1$. We have by Equation (5) that

$$p_{\mathbb{R}^+}(x_1, x_2; t) \leq t^{-3/2} x_1 x_2 e^{-(x_1-x_2)^2/(4t)} \leq t^{-3/2} x_1 x_2 e^{-\sigma^2/(4t)}, (x_1, x_2) \in C \setminus C_1.$$

Consequently,

$$\int \int_{C \setminus C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} dx_1 dx_2 \leq K t^{-3/2} e^{-\sigma^2/(4t)} \quad (11)$$

where

$$K = \int \int_C \chi_1(x_1) \chi_2(x_2) x_1^{1-\alpha_1} x_2^{1-\alpha_2} dx_1 dx_2.$$

On $C \cap \{|x_1 - x_2| \leq \epsilon/5\}$ we have that $x_2 \rightarrow \chi_2(x_2) x_2^{-\alpha_2}$ is C^∞ . Hence there exists L depending on ϵ, α_2 and on χ_2 such that $|\chi_2(x_2) x_2^{-\alpha_2} - \chi_2(1) x_1^{-\alpha_2}| \leq L|x_1 - x_2|$. It is easily seen that both $x_1 \geq \epsilon/2$ and $x_2 \geq \epsilon/2$ on $C \cap \{|x_1 - x_2| \leq \epsilon/5\}$. Since the Dirichlet heat kernel on \mathbb{R}_+ is bounded from above by $t^{-1/2}$ we have that

$$\begin{aligned} & \int \int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) x_1^{-\alpha_1} L|x_1 - x_2| dx_1 dx_2 \\ & \leq L(2/\epsilon)^{\alpha_1} t^{-1/2} \int \int_{C_1} |x_1 - x_2| dx_1 dx_2 \leq 2aL(2/\epsilon)^{\alpha_1} t^{-1/2} \sigma^2. \end{aligned} \quad (12)$$

We now choose σ^2 as to minimize $t^{-3/2} e^{-\sigma^2/(4t)} + t^{-1/2} \sigma^2$, i.e.

$$\sigma^2 = 4t \log(t^{-2}).$$

This gives that for t sufficiently small the right hand sides of Equation (11) and Equation (12) are $O(t^{1/2})$ and $O(t^{1/2} \log(t^{-1}))$ respectively. We conclude that

$$B_2 = \int \int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 + O(t^{1/2} \log(t^{-1})). \quad (13)$$

We now write

$$C_1 = (C_1 \cap \{x_1^2 \geq \epsilon^2/2\}) \cup C_1 \cap \{x_1^2 < \epsilon^2/2\} = C_2 \cup C_3.$$

Since $x_1 \geq \epsilon/2$ on C_1 we have that the integrand in the first term in the right hand side of Equation (13) is bounded by $2\epsilon^{-1}t^{-1/2}$. Hence

$$\int \int_{C_3} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \leq 2\epsilon^{-1}t^{-1/2}|C_3|,$$

where $|\cdot|$ denotes Lebesgue measure. It is easily seen that $|C_3| \leq \sigma^2/2$. Consequently,

$$0 \leq \int \int_{C_3} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \leq \epsilon^{-1}t^{-1/2}\sigma^2,$$

and so the contribution from C_3 to the integral in Equation (13) is $O(t^{1/2} \log(t^{-1}))$. Furthermore by monotonicity of the Dirichlet heat kernel

$$p_{\mathbb{R}^+}(x_1, x_2; t) \leq p_{\mathbb{R}}(x_1, x_2; t).$$

Hence

$$\begin{aligned} & \int \int_{C_2} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & \leq \int \int_{\{x_1^2 \geq \epsilon^2/2\}} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & = \int_{[\epsilon/\sqrt{2}, a/2]} \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1. \end{aligned}$$

To obtain a lower bound for the contribution from C_2 to the integral in Equation (13) we first observe that $(4\pi t)^{-1/2} e^{-(x_1+x_2)^2/(4t)} \leq t^{-1/2} e^{-\epsilon^2/(4t)}$ and $x \geq \epsilon/2$ for $(x_1, x_2) \in C_2$. Therefore

$$\begin{aligned} 0 & \leq \int \int_{C_2} (4\pi t)^{-1/2} e^{-(x_1+x_2)^2/(4t)} \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & \leq 2\epsilon^{-1}t^{-1/2} e^{-\epsilon^2/(4t)} |C_2| \leq 2a^2\epsilon^{-1}t^{-1/2} e^{-\epsilon^2/(4t)} \\ & = O(e^{-\epsilon^2/(5t)}). \end{aligned}$$

Finally

$$\begin{aligned} & \int \int_{C_2} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & \geq \int \int_{\{x^2 \geq \epsilon^2/2\}} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & \quad - \int \int_{\{|x_1-x_2| \geq \sigma\} \cap \{x_1^2 \geq \epsilon^2/2\}} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2 \\ & = \int_{[\epsilon/\sqrt{2}, a/2]} \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 \\ & \quad - \int \int_{\{|x_1-x_2| \geq \sigma\} \cap \{x_1^2 \geq \epsilon^2/2\}} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_1) x_1^{-1} dx_1 dx_2. \end{aligned}$$

Moreover

$$\begin{aligned} \int_{\{|x_1-x_2|\geq\sigma\}} p_{\mathbb{R}}(x_1, x_2; t) dx_2 &\leq \int_{\{|x_1-x_2|\geq\sigma\}} (4\pi t)^{-1/2} e^{-|x_1-x_2|\sigma/(4t)} dx_2 \\ &= 4\pi^{-1/2} t^{1/2} \sigma^{-1} e^{-\sigma^2/(4t)} = O(t^2). \end{aligned}$$

Putting all this together gives that

$$\begin{aligned} B_2 &= \int_{[\epsilon/\sqrt{2}, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx + O(t^{1/2} \log(t^{-1})) \\ &= \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} dx + 2^{-1} \log 2 + O(t^{1/2} \log(t^{-1})), \end{aligned}$$

since $\chi_1(x) \chi_2(x) x^{-1} = x^{-1}$ for $0 < x \leq \epsilon$.

In order to obtain the asymptotic behaviour of B_1 in Equation (10), we introduce polar coordinates $x = (4t)^{1/2} \rho \cos \theta$, $y = (4t)^{1/2} \rho \sin \theta$ to find that

$$\begin{aligned} B_1 &= \pi^{-1/2} \int_{[0, \pi/2]} d\theta (\cos \theta)^{-\alpha} (\sin \theta)^{\alpha-1} \\ &\quad \times \int_{[0, \epsilon/(4t)^{1/2}]} d\rho (e^{-\rho^2(1-\sin(2\theta))} - e^{-\rho^2(1+\sin(2\theta))}). \end{aligned}$$

A further change of variable $\theta = \phi + \pi/4$ yields that

$$\begin{aligned} B_1 &= (2/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \left(\frac{(\cos \phi + \sin \phi)^{\alpha-1}}{(\cos \phi - \sin \phi)^{\alpha}} + \frac{(\cos \phi - \sin \phi)^{\alpha-1}}{(\cos \phi + \sin \phi)^{\alpha}} \right) \\ &\quad \times \int_{[0, \epsilon/(4t)^{1/2}]} d\rho (e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2}) = B_3 + B_4 + B_5, \end{aligned}$$

where

$$\begin{aligned} B_3 &= (2/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \left(\frac{(\cos \phi + \sin \phi)^{\alpha-1}}{(\cos \phi - \sin \phi)^{\alpha}} + \frac{(\cos \phi - \sin \phi)^{\alpha-1}}{(\cos \phi + \sin \phi)^{\alpha}} - 2 \right) \\ &\quad \times \int_{[0, \infty)} d\rho (e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2}) \\ &= 2^{-1} \int_{[0, \pi/4]} d\phi (\cos \phi)^{-1} (\sin \phi)^{-1} \\ &\quad \times \left(\frac{(\cos \phi + \sin \phi)^{\alpha-1}}{(\cos \phi - \sin \phi)^{\alpha-1}} + \frac{(\cos \phi - \sin \phi)^{\alpha}}{(\cos \phi + \sin \phi)^{\alpha}} - 2(\cos \phi - \sin \phi) \right) \quad (14) \\ &= 2^{-1} \int_{[0, 1]} dq q^{-1} (1+q^2) \times \left\{ ((1+q)/(1-q))^{\alpha-1} \right. \\ &\quad \left. + ((1-q)/(1+q))^{\alpha} - 2(1-q)(1+q^2)^{-1/2} \right\}, \end{aligned}$$

$$\begin{aligned} B_4 &= -(2/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \\ &\quad \times \left\{ \frac{(\cos \phi + \sin \phi)^{\alpha-1}}{(\cos \phi - \sin \phi)^{\alpha}} + \frac{(\cos \phi - \sin \phi)^{\alpha-1}}{(\cos \phi + \sin \phi)^{\alpha}} - 2 \right\} \\ &\quad \times \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho (e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2}), \end{aligned}$$

and

$$B_5 = (8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[0, \epsilon/(4t)^{1/2}]} d\rho (e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2}). \quad (15)$$

We have used the standard change of variables $\tan \phi = q$ to obtain the last identity in Equation (14).

In order to find the asymptotic behaviour of B_5 as $t \downarrow 0$ we first consider the contribution of the second term in the integrand with respect to ρ in Equation (15), and write

$$\begin{aligned} & -(8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[0, \epsilon/(4t)^{1/2}]} d\rho e^{-2\rho^2 (\cos \phi)^2} \\ = & - \int_{[0, \pi/4]} d\phi (\cos \phi)^{-1} + (8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho e^{-2\rho^2 (\cos \phi)^2} \\ = & \log(2^{1/2} - 1) + O(e^{-\epsilon^2/(5t)}). \end{aligned}$$

The contribution of the first term in the integrand with respect to ρ in Equation (15) is calculated as follows:

$$\begin{aligned} & (8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[0, \epsilon/(4t)^{1/2}]} d\rho e^{-2\rho^2 (\sin \phi)^2} \\ = & (8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[0, \epsilon/(4t)^{1/2}]} d\rho e^{-2\rho^2 \phi^2} \\ & + \int_{[0, \pi/4]} d\phi ((\sin \phi)^{-1} - \phi^{-1}) \\ & + (8/\pi)^{1/2} \int_{[0, \pi/4]} d\phi \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho \left(e^{-2\rho^2 \phi^2} - e^{-2\rho^2 (\sin \phi)^2} \right). \end{aligned} \tag{16}$$

The third term in the right hand side of Equation (16) is $O(e^{-\epsilon^2/(5t)})$. The second term in the right hand side of Equation (16) is equal to $\log(2^{1/2} - 1) + 3 \log 2 - \log \pi$. The first term in the right hand side of Equation (16) equals

$$\begin{aligned} & (4/\pi)^{1/2} \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\phi \int_{[0, \phi]} d\rho e^{-\rho^2} \\ = & (4/\pi)^{1/2} \left(\log(\pi\epsilon/(32t)^{1/2}) \right) \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\rho e^{-\rho^2} \\ & - (4/\pi)^{1/2} \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\phi (\log \phi) e^{-\phi^2} \\ = & 2^{-1} \log(\epsilon^2/t) + \log \pi + 2^{-1}\gamma - 3 \cdot 2^{-1} \log 2 + O(e^{-\epsilon^2/(5t)}), \end{aligned}$$

where we have used Equation (4.333) in [18] together with

$$\int_{[0, \pi\epsilon/(32t)^{1/2}]} d\phi (\log \phi) e^{-\phi^2} = \int_{[0, \infty)} d\phi (\log \phi) e^{-\phi^2} + O(e^{-\epsilon^2/(5t)}).$$

We find that

$$B_5 = 2^{-1} \log(\epsilon^2/t) + 2^{-1}\gamma + 2 \log(2^{1/2} - 1) + 3 \cdot 2^{-1} \log 2 + O(e^{-\epsilon^2/(5t)}).$$

In order to estimate B_4 we first note that by expanding $\sin \phi$ and $\cos \phi$ around 0 we have that

$$\frac{(\cos \phi + \sin \phi)^{\alpha-1}}{(\cos \phi - \sin \phi)^\alpha} + \frac{(\cos \phi - \sin \phi)^{\alpha-1}}{(\cos \phi + \sin \phi)^\alpha} - 2 = O(\phi^2).$$

Furthermore for $\phi \in [0, \pi/4]$,

$$\begin{aligned}
0 &\leq \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho \left(e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2} \right) \\
&\leq (4t)^{1/2} \epsilon^{-1} \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho \left(e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2} \right) \\
&\leq (4t)^{1/2} \epsilon^{-1} \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho \rho \left(e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2} \right) \\
&\leq (4t)^{1/2} \epsilon^{-1} \int_{[0, \infty)} d\rho \rho \left(e^{-2\rho^2(\sin \phi)^2} - e^{-2\rho^2(\cos \phi)^2} \right) \\
&= t^{1/2} \epsilon^{-1} \left((\sin \phi)^{-2} - (\cos \phi)^{-2} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
|B_4| &\leq \pi^{-1/2} t^{1/2} \epsilon^{-1} \int_{[0, \pi/4]} d\phi (\sin \phi)^{-2} (\cos \phi)^{-2} \\
&\times \left| \frac{(\cos \phi + \sin \phi)^\alpha}{(\cos \phi - \sin \phi)^{\alpha-1}} + \frac{(\cos \phi - \sin \phi)^\alpha}{(\cos \phi + \sin \phi)^{\alpha-1}} - 2((\cos \phi)^2 - (\sin \phi)^2) \right|.
\end{aligned}$$

We see that the integral with respect to ϕ converges both at $\phi = 0$ and at $\phi = \pi/4$. We conclude that $B_4 = O(t^{1/2})$. \square

3. THE PROOF OF THEOREM 3

We shall assume that $\Re(\alpha_1) \ll 0$ and $\Re(\alpha_2) \ll 0$ and then apply analytic continuation to establish the general case. We shall also assume that $\alpha_1 + \alpha_2 \notin \mathbb{Z}$ to ensure that the interior and the boundary terms do not interact. The invariants $\beta_j^{\partial M}$ are given by local formula. Standard arguments using dimensional analysis yield the following result; as these arguments are by now standard (see, for example, the discussion in [5]), we omit details in the interests of brevity.

Lemma 7. *There exist universal constants $\varepsilon_{\alpha_1, \alpha_2}^i$ so that:*

$$\begin{aligned}
\beta_0^{\partial M} &= \int_{\partial M} \varepsilon_{\alpha_1, \alpha_2}^0 \langle \psi_1^0, \psi_2^0 \rangle dy, \\
\beta_1^{\partial M} &= \int_{\partial M} \{ \varepsilon_{\alpha_1, \alpha_2}^1 \langle \psi_1^1, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^2 \langle L_{aa} \psi_1^0, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^3 \langle \psi_1^0, \psi_2^1 \rangle \} dy, \\
\beta_2^{\partial M} &= \int_{\partial M} \{ \varepsilon_{\alpha_1, \alpha_2}^4 \langle \psi_1^2, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^5 \langle L_{aa} \psi_1^1, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^6 \langle E \psi_1^0, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^7 \langle \psi_1^0, \psi_2^2 \rangle \\
&\quad + \varepsilon_{\alpha_1, \alpha_2}^8 \langle L_{aa} \psi_1^0, \psi_2^1 \rangle + \varepsilon_{\alpha_1, \alpha_2}^9 \langle \text{Ric}_{mm} \psi_1^0, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^{10} \langle L_{aa} L_{bb} \psi_1^0, \psi_2^0 \rangle \\
&\quad + \varepsilon_{\alpha_1, \alpha_2}^{11} \langle L_{ab} L_{ab} \psi_1^0, \psi_2^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^{12} \langle \psi_{1:a}^0, \psi_{2:a}^0 \rangle + \varepsilon_{\alpha_1, \alpha_2}^{13} \langle \tau \psi_1^0, \psi_2^0 \rangle \\
&\quad + \varepsilon_{\alpha_1, \alpha_2}^{14} \langle \psi_1^1, \psi_2^1 \rangle \} dy.
\end{aligned}$$

Remark 2. We note that $\varepsilon_{\alpha_1, \alpha_2}^0 = c_{\alpha_1, \alpha_2}$ is given by Equation (2).

There is a basic symmetry which is useful. Let e^{-tD} denote the fundamental solution of the Dirichlet Laplacian and let \tilde{D} be the dual operator on the dual vector bundle \tilde{V} . The lemma below follows immediately from the identity

$$Q(\psi_1, \psi_2, D)(t) = \int_M \langle e^{-tD} \psi_1, \psi_2 \rangle dx = \int_M \langle \psi_1, e^{-t\tilde{D}} \psi_2 \rangle dx = Q(\psi_2, \psi_1, \tilde{D})(t).$$

Lemma 8. *Adopt the notation of Lemma 7.*

$$\begin{aligned}
\varepsilon_{\alpha_1, \alpha_2}^0 &= \varepsilon_{\alpha_2, \alpha_1}^0, & \varepsilon_{\alpha_1, \alpha_2}^1 &= \varepsilon_{\alpha_2, \alpha_1}^3, & \varepsilon_{\alpha_1, \alpha_2}^2 &= \varepsilon_{\alpha_2, \alpha_1}^2, & \varepsilon_{\alpha_1, \alpha_2}^4 &= \varepsilon_{\alpha_2, \alpha_1}^7, \\
\varepsilon_{\alpha_1, \alpha_2}^5 &= \varepsilon_{\alpha_2, \alpha_1}^8, & \varepsilon_{\alpha_1, \alpha_2}^6 &= \varepsilon_{\alpha_2, \alpha_1}^6, & \varepsilon_{\alpha_1, \alpha_2}^9 &= \varepsilon_{\alpha_2, \alpha_1}^9, & \varepsilon_{\alpha_1, \alpha_2}^{10} &= \varepsilon_{\alpha_2, \alpha_1}^{10}, \\
\varepsilon_{\alpha_1, \alpha_2}^{11} &= \varepsilon_{\alpha_2, \alpha_1}^{11}, & \varepsilon_{\alpha_1, \alpha_2}^{12} &= \varepsilon_{\alpha_2, \alpha_1}^{12}, & \varepsilon_{\alpha_1, \alpha_2}^{13} &= \varepsilon_{\alpha_2, \alpha_1}^{13}, & \varepsilon_{\alpha_1, \alpha_2}^{14} &= \varepsilon_{\alpha_2, \alpha_1}^{14}.
\end{aligned}$$

Next, we consider some product formulae:

Lemma 9. Suppose that $M = M_1 \times M_2$, that $g_M = g_{M_1} + g_{M_2}$, that $\partial M_1 = \emptyset$, and that $D_M = D_{M_1} + D_{M_2}$ where D_{M_1} and D_{M_2} are scalar operators of Laplace type on M_1 and on M_2 , respectively. Suppose that $\psi_1^M = \psi_1^{M_1} \psi_1^{M_2}$ and $\psi_2^M = \psi_2^{M_1} \psi_2^{M_2}$ decompose similarly. Then

- (a) $\beta(\psi_1^M, \psi_2^M, D_M)(t) = \beta(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1})(t) \cdot \beta(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2})(t)$.
- (b) $\int_{\partial M} \beta_{k,\alpha_1,\alpha_2}^{\partial M}(\psi_1^M, \psi_2^M, D_M) dy = \sum_{2n+j=k} \frac{(-1)^n}{n!} \int_{M_1} \langle \psi_1^{M_1}, (\tilde{D}_{M_1})^n \psi_2^{M_1} \rangle dx_{M_1} \times \int_{\partial M_2} \beta_{j,\alpha_1,\alpha_2}^{\partial M_2}(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2}) dy_{M_2}$.
- (c) The universal constants $\varepsilon_{\alpha_1,\alpha_2}^i$ are dimension free.
- (d) $\varepsilon_{\alpha_1,\alpha_2}^6 = \varepsilon_{\alpha_1,\alpha_2}^0$, $\varepsilon_{\alpha_1,\alpha_2}^{13} = 0$, and $\varepsilon_{\alpha_1,\alpha_2}^{12} = -\varepsilon_{\alpha_1,\alpha_2}^0$.

Proof. Assertion (a) follows from the identity $e^{-tD_M} = e^{-tD_{M_1}} e^{-tD_{M_2}}$ and Assertion (b) follows from Assertion (a). If we take $M_1 = S^1$, $D_{M_1} = -\partial_\theta^2$, $\psi_1^{M_1} = 1$, and $\psi_2^{M_1} = 1$, we have that $\beta(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1})(t) = 2\pi$. This then yields the identity

$$\int_{\partial M} \beta_{k,\alpha_1,\alpha_2}^{\partial M}(\psi_1^{M_2}, \psi_2^{M_2}, D) dy = 2\pi \int_{\partial M_2} \beta_{k,\alpha_1,\alpha_2}^{\partial M_2}(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2}) dy_2.$$

Assertion (c) now follows. We take $M_2 = [0, 1]$ and $D_2 = -\partial_r^2$. We take

$$\begin{aligned} \psi_1^{M_2} &= \psi_2^{M_2} = 0 \quad \text{near } r = 1, \\ \psi_2^{M_2} &= r^{-\alpha_2} \quad \text{and} \quad \psi_1^{M_2} = r^{-\alpha_1} \quad \text{near } r = 0. \end{aligned}$$

Since the structures on M_2 are flat, we have $\psi_1^k = \psi_2^k = 0$ for $k > 0$ while

$$\psi_2^0 = \psi_1^0 = \begin{cases} 0 & \text{at } r = 1 \\ 1 & \text{at } r = 0 \end{cases}.$$

Consequently,

$$\beta_k^{\partial M_2}(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2})(r) = \begin{cases} 0 & \text{if } r = 1 \text{ and } k \geq 0 \\ 0 & \text{if } r = 0 \text{ and } k > 0 \\ \varepsilon_{\alpha_1,\alpha_2}^0 & \text{if } r = 0 \text{ and } k = 0 \end{cases}.$$

As the second fundamental form vanishes, the distinction between ‘;’ and ‘.’ disappears, and we have $\tilde{D}_1 \psi_2^{M_1} = -(\psi_{2;aa}^{M_1} + \tilde{E} \psi_2^{M_1})$. Calculating on the interior then implies that

$$\beta_2(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1}) = \int_{M_1} \langle \psi_1^{M_1}, \psi_{2;aa}^{M_1} + \tilde{E} \psi_2^{M_1} \rangle dx_1.$$

We may therefore use Assertion (b) to derive the following identity from which Assertion (e) will follow:

$$\int_{\partial M} \beta_{2,\alpha_1,\alpha_2}^{\partial M}(\psi_1^M, \psi_2^M, D_M) dy = \varepsilon_{\alpha_1,\alpha_2}^0 \int_{M_1} \langle \psi_1^{M_1}, \psi_{2;aa}^{M_1} + \tilde{E} \psi_2^{M_1} \rangle dx_1. \quad \square$$

We continue our study by index shifting:

Lemma 10.

$$\begin{aligned} \varepsilon_{\alpha_1,\alpha_2}^1 &= \varepsilon_{\alpha_1-1,\alpha_2}^0, & \varepsilon_{\alpha_1,\alpha_2}^4 &= \varepsilon_{\alpha_1-2,\alpha_2}^0, & \varepsilon_{\alpha_1,\alpha_2}^5 &= \varepsilon_{\alpha_1-1,\alpha_2}^2, \\ \varepsilon_{\alpha_1,\alpha_2}^3 &= \varepsilon_{\alpha_1,\alpha_2-1}^0, & \varepsilon_{\alpha_1,\alpha_2}^7 &= \varepsilon_{\alpha_1,\alpha_2-2}^0, & \varepsilon_{\alpha_1,\alpha_2}^8 &= \varepsilon_{\alpha_1,\alpha_2-1}^2, \\ \varepsilon_{\alpha_1,\alpha_2}^{14} &= \varepsilon_{\alpha_1-1,\alpha_2-1}^0. \end{aligned}$$

Proof. We assume ψ_1 and ψ_2 have compact support near the boundary of M . We set $\tilde{\psi}_1 := (\delta^{n_1} \psi) \delta^{-\alpha_1-n_1}$ and $\tilde{\psi}_2 := (\delta^{n_2} \psi_2) \delta^{-\alpha_2-n_2}$ for $n_i \in \mathbb{N}$. We compute:

$$\begin{aligned} & \sum_k t^{(1+k-n_1-\alpha_1-n_2-\alpha_2)/2} \int_{\partial M} \beta_{k,n_1+\alpha_1,n_2+\alpha_2}(\tilde{\psi}_1, \tilde{\psi}_2, D) dy \\ & \sim \sum_\ell t^{(1+\ell-\alpha_1-\alpha_2)/2} \int_{\partial M} \beta_{\ell,\alpha_1,\alpha_2}(\psi_1, \psi_2, D) dy. \end{aligned}$$

We set $k = \ell + n_1 + n_2$ and equate powers of t to see

$$\beta_{\ell+n_1+n_2, n_1+\alpha_1, n_2+\alpha_2}(\tilde{\psi}_1, \tilde{\psi}_2, D) = \beta_{\ell, \alpha_1, \alpha_2}(\psi_1, \psi_2, D).$$

Note that $\tilde{\psi}_1^{\mu+n_1} = \psi_1^\mu$ and $\tilde{\psi}_2^{\nu+n_2} = \psi_2^\nu$. The desired result now follows by taking $(n_1, n_2) = (1, 0), (0, 1), (2, 0), (1, 1)$, and $(0, 2)$. \square

Lemma 11. *Let \mathbb{T}^{m-1} denote the torus with periodic parameters (y_1, \dots, y_{m-1}) and let $M := \mathbb{T}^{m-1} \times [0, 1]$. Let $f_a \in C^\infty([0, 1])$ have compact support near $r = 0$ with $f_a(0) = 0$. Let $\Theta(r) \in C^\infty([0, 1])$ have compact support near $r = 0$ with $\Theta \equiv 1$ near $r = 0$. Let $\delta_a \in \mathbb{R}$. Set*

$$ds_M^2 = \sum_a e^{2f_a(r)} dy_a \circ dy_a + dr \circ dr, \quad \psi_2 := \Theta(r) e^{-\sum_a f_a(r) r^{-\alpha_2}},$$

$$D_M := -\sum_a e^{-2f_a(r)} (\partial_{y_a}^2 + \delta_a \partial_{y_a}) - \partial_r^2, \quad \psi_1 := \Theta(r) r^{-\alpha_1}.$$

- (a) *If $k > 0$, then $\int_{\partial M} \beta_{k, \alpha_1, \alpha_2}^{\partial M}(\psi_1, \psi_2, D_M) dy = 0$.*
- (b) $-\frac{1}{2} \varepsilon_{\alpha_1, \alpha_2}^1 - \varepsilon_{\alpha_1, \alpha_2}^2 - \frac{1}{2} \varepsilon_{\alpha_1, \alpha_2}^3 = 0$.
- (c) $-\frac{1}{4} (\varepsilon_{\alpha_1, \alpha_2}^6 + \varepsilon_{\alpha_1, \alpha_2}^{12}) = 0$.
- (d) $-\frac{1}{4} \varepsilon_{\alpha_1, \alpha_2}^4 + \frac{1}{2} \varepsilon_{\alpha_1, \alpha_2}^6 - \frac{1}{4} \varepsilon_{\alpha_1, \alpha_2}^7 - \varepsilon_{\alpha_1, \alpha_2}^9 = 0$.
- (e) $\frac{1}{8} \varepsilon_{\alpha_1, \alpha_2}^4 + \frac{1}{2} \varepsilon_{\alpha_1, \alpha_2}^5 + \frac{1}{4} \varepsilon_{\alpha_1, \alpha_2}^6 + \frac{1}{8} \varepsilon_{\alpha_1, \alpha_2}^7 + \frac{1}{2} \varepsilon_{\alpha_1, \alpha_2}^8 + \varepsilon_{\alpha_1, \alpha_2}^{10} + \frac{1}{4} \varepsilon_{\alpha_1, \alpha_2}^{14} = 0$.
- (f) $-\varepsilon_{\alpha_1, \alpha_2}^9 + \varepsilon_{\alpha_1, \alpha_2}^{11} = 0$.

Proof. We use $-\partial_r^2$ on $[0, 1]$ and D_M on M . Since Θ vanishes near $r = 1$, this boundary component plays no role. Let $u(r; t)$ be the solution of the heat equation on $[0, 1]$ with Dirichlet boundary conditions and initial temperature ψ_1 . The parameter r is the geodesic distance to the boundary near $r = 0$. Since the problem decouples, $u(r; t)$ is also the solution of the heat equation on M with Dirichlet boundary conditions. The Riemannian measure

$$dx = \sqrt{\det g_{ij}} dy dr = e^{\sum_a f_a} dy dr.$$

As $\psi_2 = \Theta e^{-\sum_a f_a r^{-\alpha_2}}$, $\psi_2 dx = \Theta r^{-\alpha_2} dy dr$. Since $\text{vol}(\mathbb{T}^{m-1}) = (2\pi)^{m-1}$,

$$\begin{aligned} Q(\psi_1, \psi_2, D)(t) &= \int u(r; t) \psi_2 dx = (2\pi)^{m-1} \int_0^1 u(r; t) \Theta(r) r^{-\alpha_2} dr \\ &= (2\pi)^{m-1} \beta(\Theta r^{-\alpha_1}, \Theta r^{-\alpha_2}, -\partial_r^2)(t). \end{aligned}$$

The structures are flat on $[0, 1]$. Since Θ vanishes identically near $r = 1$ and Θ is identically 1 near $r = 0$, only the term β_0 is relevant in computing the boundary terms; the $\beta_{k, \alpha_1, \alpha_2}$ vanish for $k \geq 1$.

To apply Assertion (a), we must determine the relevant tensors. We have:

$$\begin{aligned} \Gamma_{abm} &= -f'_a \delta_{ab} e^{2f_a}, & \Gamma_{ab}^m &= -f'_a e^{2f_a} \delta_{ab}, \\ \Gamma_{amb} &= f'_a \delta_{ab} e^{2f_a}, & \Gamma_{am}^b &= f'_a \delta_{a,b}, \\ L_{ab} &= \Gamma_{ab}^m|_{\partial M} = -f'_a \delta_{ab}, \\ \omega_a &= \frac{1}{2} e^{2f_a} \delta_a, & \tilde{\omega}_a &= -\omega_a = -\frac{1}{2} e^{2f_a} \delta_a, \\ \omega_m &= -\frac{1}{2} \sum_a f'_a, & \tilde{\omega}_m &= -\omega_m = \frac{1}{2} \sum_a f'_a. \end{aligned}$$

Consequently:

$$\begin{aligned} R_{ambm} &= g((\nabla_a \nabla_m - \nabla_m \nabla_a) e_b, e_m) = \Gamma_{ac}^m \Gamma_{mb}^c - \partial_m \Gamma_{ab}^m \\ &= \{-(f'_a)^2 + f''_a + 2(f'_a)^2\} e^{2f_a} \delta_{ab}, \\ \text{Ric}_{mm} &= -\sum_a \{f''_a + (f'_a)^2\}, \\ E|_{\partial M} &= -\partial_m \omega_m - \omega_a^2 - \omega_m^2 + \omega_m \Gamma_{aa}^m \\ &= \frac{1}{2} \sum_a f''_a - \frac{1}{4} \sum_a \delta_a^2 - \frac{1}{4} \sum_{a,b} f'_a f'_b + \frac{1}{2} \sum_{a,b} f'_a f'_b \end{aligned}$$

$$= \frac{1}{2} \sum_a f_a'' - \frac{1}{4} \sum_a \delta_a^2 + \frac{1}{4} \sum_{a,b} f_a' f_b'.$$

We compute:

$$\begin{aligned} \psi_1^0 &= 1, \\ \psi_1^1 &= \{\nabla_{\partial r}(r^\alpha \psi_1)\}|_{\partial M} = \{(\partial_r - \frac{1}{2} \sum_a f_a')(1)\}|_{\partial M} = -\frac{1}{2} \sum_a f_a', \\ \psi_1^2 &= \frac{1}{2} \{(\nabla_{\partial r})^2(r^\alpha \psi_1)\}|_{\partial M} = \frac{1}{2} \{(\partial_r - \frac{1}{2} \sum_a f_a')^2(1)\}|_{\partial M} \\ &= \frac{1}{8} \sum_{a,b} f_a' f_b' - \frac{1}{4} \sum_a f_a'', \\ \psi_2^0 &= 1, \\ \psi_2^1 &= \{\tilde{\nabla}_{\partial r}(\psi_2)\}|_{\partial M} = \{(\partial_r + \frac{1}{2} \sum_a f_a')(e^{-\sum_a f_a})\}|_{\partial M} = -\frac{1}{2} \sum_a f_a', \\ \psi_2^2 &= \frac{1}{2} \{(\tilde{\nabla}_{\partial r})^2 \psi_2\}|_{\partial M} = \frac{1}{2} \{(\partial_r + \frac{1}{2} \sum_a f_a')^2(e^{-\sum_a f_a})\}|_{\partial M} \\ &= \frac{1}{8} \sum_{a,b} f_a' f_b' - \frac{1}{4} \sum_a f_a''. \end{aligned}$$

Considering the term $\sum_a f_a'$ in $\beta_{1,\alpha}^{\partial M}$ yields Assertion (b), considering the term $\sum_a \delta_a^2$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (c), considering the term $\sum_a f_a''$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (d), considering the term $\sum_{a,b} f_a' f_b'$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (e), and considering the term $\sum_a (f_a')^2$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (f).

3.1. The Proof of Theorem 3. We must now simply trace through the logic train. We have computed that:

$$\begin{aligned} \varepsilon_{\alpha_1, \alpha_2} &= c_{\alpha_1, \alpha_2}, \\ \varepsilon_{\alpha_1, \alpha_2}^1 &= c_{\alpha_1-1, \alpha_2} \text{ and } \varepsilon_{\alpha_1, \alpha_2}^3 = c_{\alpha_1, \alpha_2-1}, \\ \varepsilon_{\alpha_1, \alpha_2}^2 &= -\frac{1}{2}(\varepsilon_{\alpha_1, \alpha_2}^1 + \varepsilon_{\alpha_1, \alpha_2}^3) = -\frac{1}{2}(c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1}), \\ \varepsilon_{\alpha_1, \alpha_2}^4 &= c_{\alpha_1-2, \alpha_2} \text{ and } \varepsilon_{\alpha_1, \alpha_2}^7 = c_{\alpha_1, \alpha_2-2}, \\ \varepsilon_{\alpha_1, \alpha_2}^6 &= c_{\alpha_1, \alpha_2} \text{ and } \varepsilon_{\alpha_1, \alpha_2}^{14} = c_{\alpha_1-1, \alpha_2-1}, \\ \varepsilon_{\alpha_1, \alpha_2}^{12} &= -\varepsilon_{\alpha_1, \alpha_2}^6 = -c_{\alpha_1, \alpha_2}, \\ \varepsilon_{\alpha_1, \alpha_2}^5 &= \varepsilon_{\alpha_1-1, \alpha_2}^2 = -\frac{1}{2}(c_{\alpha_1-2, \alpha_2} + c_{\alpha_1-1, \alpha_2-1}), \\ \varepsilon_{\alpha_1, \alpha_2}^8 &= \varepsilon_{\alpha_1, \alpha_2-1}^2 = -\frac{1}{2}(c_{\alpha_1-1, \alpha_2-1} + c_{\alpha_1, \alpha_2-2}), \\ \varepsilon_{\alpha_1, \alpha_2}^{11} &= \varepsilon_{\alpha_1, \alpha_2}^9 = -\frac{1}{4}\varepsilon_{\alpha_1, \alpha_2}^4 - \frac{1}{4}\varepsilon_{\alpha_1, \alpha_2}^7 + \frac{1}{2}\varepsilon_{\alpha_1, \alpha_2}^6 \\ &= -\frac{1}{4}c_{\alpha_1-2, \alpha_2} - \frac{1}{4}c_{\alpha_1, \alpha_2-2} + \frac{1}{2}c_{\alpha_1, \alpha_2}, \\ \varepsilon_{\alpha_1, \alpha_2}^{10} &= -\left\{\frac{1}{8}\varepsilon_{\alpha_1, \alpha_2}^4 + \frac{1}{2}\varepsilon_{\alpha_1, \alpha_2}^5 + \frac{1}{4}\varepsilon_{\alpha_1, \alpha_2}^6 + \frac{1}{8}\varepsilon_{\alpha_1, \alpha_2}^7 + \frac{1}{2}\varepsilon_{\alpha_1, \alpha_2}^8 + \frac{1}{4}\varepsilon_{\alpha_1, \alpha_2}^{14}\right\} \\ &= -\left\{\frac{1}{8}c_{\alpha_1-2, \alpha_2} - \frac{1}{4}(c_{\alpha_1-2, \alpha_2} + c_{\alpha_1-1, \alpha_2-1}) + \frac{1}{4}c_{\alpha_1, \alpha_2}\right. \\ &\quad \left.+ \frac{1}{8}c_{\alpha_1, \alpha_2-2} - \frac{1}{4}(c_{\alpha_1-1, \alpha_2-1} + c_{\alpha_1, \alpha_2-2}) + \frac{1}{4}c_{\alpha_1-1, \alpha_2-1}\right\} \\ &= \left\{(-\frac{1}{8} + \frac{1}{4})c_{\alpha_1-2, \alpha_2} + (\frac{1}{4} - \frac{1}{4} + \frac{1}{4})c_{\alpha_1-1, \alpha_2-1} + (-\frac{1}{8} + \frac{1}{4})c_{\alpha_1, \alpha_2-2}\right. \\ &\quad \left.- \frac{1}{4}c_{\alpha_1, \alpha_2}\right\}. \end{aligned} \quad \square$$

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